



ELSEVIER

Discrete Mathematics 251 (2002) 77–85

DISCRETE  
MATHEMATICS

www.elsevier.com/locate/disc

# A note on hamiltonicity of generalized net-free graphs of large diameter

Jan Brousek<sup>a,\*</sup>, Ralph J. Faudree<sup>b</sup>, Zdeněk Ryjáček<sup>c,1</sup>

<sup>a</sup>*Department of Mathematics, University of West Bohemia, P.O. Box 314 306 14 Pilsen  
Czech Republic*

<sup>b</sup>*Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA*

<sup>c</sup>*Department of Mathematics, University of West Bohemia, 306 14 Pilsen, Czech Republic*

Received 14 June 1999; revised 6 May 2000; accepted 3 June 2001

## Abstract

A generalized  $(i, j, k)$ -net  $N_{i,j,k}$  is the graph obtained by identifying each of the vertices of a triangle with an endvertex of one of three vertex-disjoint paths of lengths  $i, j, k$ . We prove that every 2-connected claw-free  $N_{1,2,j}$ -free graph of diameter at least  $\max\{7, 2j\}$  ( $j \geq 2$ ) is hamiltonian. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 05C45

Keywords: Hamiltonian graphs; Forbidden subgraphs; Claw-free graphs

## 1. Introduction

In this paper, we consider finite simple undirected graphs  $G = (V(G), E(G))$  and for concepts and notations not defined here we refer the reader to [2].

For a set  $S \subset V(G)$  we denote by  $N(S)$  the *neighborhood* of  $S$ , i.e. the set of all vertices of  $G$  which have a neighbor in  $S$ . If  $S = \{x\}$ , we simply write  $N(x)$  for  $N(\{x\})$ . For any subset  $M \subset V(G)$ , we denote  $N_M(S) = N(S) \cap M$ . If  $H$  is a subgraph of  $G$ , we write  $N_H(S)$  for  $N_{V(H)}(S)$ . For subsets  $M, N \subset V(G)$ ,  $M \cap N = \emptyset$ , we denote  $E(M, N) = \{xy \in E(G) \mid x \in M, y \in N\}$ . The induced subgraph on a set  $M \subset V(G)$  in  $G$  will be denoted by  $\langle M \rangle_G$ .

We denote by  $\text{diam}(G)$  the *diameter* of  $G$ , i.e. the largest distance of a pair of vertices  $x, y \in V(G)$ . A path with endvertices  $x, y$  will be sometimes referred to as an  $xy$ -path. If  $x, z$  are vertices at distance  $\text{diam}(G)$ , then any shortest  $xz$ -path will be called

\* Corresponding author.

E-mail addresses: brousek@kma.zcu.cz (J. Brousek), rfaudree@cc.memphis.edu (R.J. Faudree), ryjacek@kma.zcu.cz (Z. Ryjáček).

<sup>1</sup> Research supported by grant GA ČR No. 201/97/0407.

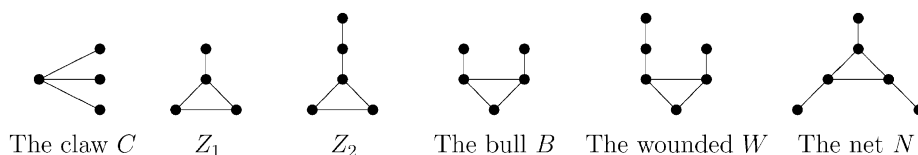


Fig. 1.

a *diameter path* of  $G$ . By  $c(G)$  we denote the *circumference* of  $G$ , i.e. the length of a longest cycle in  $G$ . A graph  $G$  is *hamiltonian* if  $c(G) = |V(G)|$ .

If  $H_1, \dots, H_k$  ( $k \geq 1$ ) are graphs, then a graph  $G$  is said to be  $H_1 H_2 \dots H_k$ -free if  $G$  contains no copy of any of the graphs  $H_1, \dots, H_k$  as an induced subgraph. The graphs  $H_1, \dots, H_k$  will also be referred to in this context as *forbidden subgraphs*. Specifically, the graph  $K_{1,3}$  will also be denoted by  $C$  and called the *claw* and in this case we say that  $G$  is *claw-free*. Whenever vertices of an induced claw are listed, its *center* (i.e. its unique vertex of degree 3) is always the first vertex of the list. We denote by  $P_i$  the path on  $i$  vertices. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1.

We say that a vertex  $x \in V(G)$  is *locally connected* if  $\langle N(x) \rangle_G$  is a connected graph. A locally connected vertex with a noncomplete neighborhood is called *eligible* and the graph  $G'_x$ , obtained from  $G$  by adding to the neighborhood of an eligible vertex  $x$  all missing edges, (i.e. such that  $N(x)$  induces a complete graph in  $G'_x$ ), is called the *local completion of  $G$  at  $x$* . The following was proved in [12].

**Theorem A** (Ryjáček [12]). *Let  $G$  be a claw-free graph and let  $x$  be an eligible vertex of  $G$ . Let  $N'_x = \{uv \mid u, v \in N_G(x), uv \notin E(G)\}$  and let  $G'_x$  be the graph with vertex set  $V(G'_x) = V(G)$  and with edge set  $E(G'_x) = E(G) \cup N'_x$ . Then*

- (i) *the graph  $G'_x$  is claw-free,*
- (ii)  *$c(G'_x) = c(G)$ .*

It can be shown that a graph which is obtained from a claw-free graph  $G$  by recursively performing the local completion operation, as long as there is at least one eligible vertex, is uniquely determined and is again claw-free. This graph is called the *closure of  $G$*  and is denoted by  $\text{cl}(G)$ . The following theorem summarizes the basic properties of the closure.

**Theorem B** (Ryjáček [12]). *Let  $G$  be a claw-free graph. Then*

- (i)  *$\text{cl}(G)$  is well-defined (i.e., uniquely determined by  $G$ ),*
- (ii) *there is a triangle-free graph  $H$  such that  $\text{cl}(G)$  is the line graph of  $H$ ,*
- (iii)  *$c(\text{cl}(G)) = c(G)$ ,*
- (iv)  *$G$  is hamiltonian if and only if  $\text{cl}(G)$  is hamiltonian.*

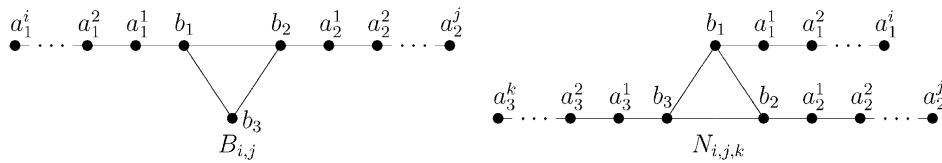


Fig. 2.

If  $G = \text{cl}(G)$ , then we say that the graph  $G$  is *closed* (thus,  $G$  is closed if and only if  $G$  is the line graph of a triangle-free graph).

There are many results dealing with hamiltonian properties in classes of graphs defined in terms of forbidden induced subgraphs (see e.g. [3,4,7,10,11]). Bedrossian [1] (see also [8]) characterized all pairs  $X, Y$  of connected forbidden subgraphs implying hamiltonicity.

**Theorem C** (Bedrossian [1]). *Let  $X$  and  $Y$  be connected graphs with  $X, Y \neq P_3$ , and let  $G$  be a 2-connected graph that is not a cycle. Then,  $G$  being  $XY$ -free implies that  $G$  is hamiltonian if and only if (up to symmetry)  $X = C$  and  $Y = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$  or  $W$ .*

The classes of  $CB$ -free graphs and  $CN$ -free graphs were extended in [5] as follows. We denote by (see also Fig. 2):

$B_{i,j}$  ( $j \geq i \geq 1$ ) —the *generalized  $(i,j)$ -bull*, i.e. the graph which is obtained by identifying each of some two distinct vertices of a triangle with an endvertex of one of two vertex-disjoint paths of lengths  $i, j$ ,

$N_{i,j,k}$  ( $k \geq j \geq i \geq 1$ ) —the *generalized  $(i,j,k)$ -net*, i.e. the graph which is obtained by identifying each vertex of a triangle with an endvertex of one of three vertex-disjoint paths of lengths  $i, j, k$ .

Thus,  $B_{1,1} \simeq B$ ,  $B_{1,2} \simeq W$  and  $N_{1,1,1} \simeq N$ . The graphs  $Z_i$  ( $i \geq 1$ ) can be defined analogously (see Fig. 1). We further denote by  $H$  the *hourglass*, i.e. the graph consisting of two triangles with a common vertex.

A class  $\mathcal{C}$  of graphs such that  $\text{cl}(G) \in \mathcal{C}$  for every  $G \in \mathcal{C}$  is called a *stable class*. Clearly, the class of  $CA$ -free graphs is trivially stable if  $A$  is not claw-free or if  $A$  is not closed. The following theorem characterizes all connected closed claw-free graphs  $A$  for which the  $CA$ -free class is stable.

**Theorem D** (Brousek et al. [6]). *Let  $A$  be a closed connected claw-free graph. Then  $G$  being  $CA$ -free implies  $\text{cl}(G)$  is  $CA$ -free if and only if*

$$A \in \{H, C_3\} \cup \{P_i \mid i \geq 3\} \cup \{Z_i \mid i \geq 1\} \cup \{N_{i,j,k} \mid i, j, k \geq 1\}.$$

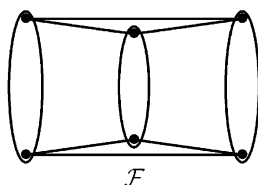


Fig. 3.

Making use of the special structure of closed claw-free graphs, the results on hamiltonicity in  $CP_i$ -free,  $CZ_i$ -free and  $CN$ -free graphs were extended in [5] and [6] to larger classes of  $CP_7$ -free,  $CZ_4$ -free and  $CN_{1,2,2}N_{1,1,3}$ -free graphs by characterizing the classes of nonhamiltonian exceptions. Specifically, the following was proved in [5] ( $\mathcal{F}$  denotes the class of graphs shown in Fig. 3, where the elliptical parts represent cliques of arbitrary order).

**Theorem E** (Brousek et al. [5]). *If  $G$  is a 2-connected  $CN_{1,1,2}$ -free graph, then either  $G$  is hamiltonian or  $G \in \mathcal{F}$ .*

In the class of  $CB_{i,j}$ -free graphs, the following result was proved in [9].

**Theorem F** (Faudree et al. [9]). *Let  $j \geq 2$  be an integer and let  $G$  be a 2-connected  $CB_{2,j}$ -free graph of diameter  $d \geq \max\{7, 2j\}$ . Then  $G$  is hamiltonian.*

In the main result of this paper, we extend this result to the class of all 2-connected  $CN_{1,2,j}$ -free graphs.

## 2. Main result

It is easy to see that, for any  $i, j, k \geq 1$ , there are  $CN_{i,j,k}$ -free graphs of arbitrarily large diameter. A simple example can be obtained by taking  $d + 1$  vertex-disjoint cliques  $K_0, K_1, \dots, K_d$  (for sufficiently large  $d$ ) and by adding all of the edges between consecutive cliques, namely  $\{xy \mid x \in K_i, y \in K_{i+1}, i = 0, 1, \dots, d - 1\}$ .

In the main result of this paper, Theorem 1, we show that nonhamiltonian 2-connected  $CN_{1,2,j}$ -free graphs must have small diameter.

**Theorem 1.** *Let  $j \geq 2$  be an integer and let  $G$  be a 2-connected  $CN_{1,2,j}$ -free graph of diameter  $d \geq \max\{7, 2j\}$ . Then  $G$  is hamiltonian.*

Since every  $B_{2,j}$ -free or  $N_{1,1,j}$ -free graph is also  $N_{1,2,j}$ -free, Theorem 1 implies as immediate corollaries Theorem F and a corresponding result for  $N_{1,1,j}$ -free graphs. However, while it is shown in [9] that the assumptions of Theorem F are sharp, the lower bound on the diameter of  $G$  can be slightly improved in the case of  $N_{1,1,j}$ -free graphs.

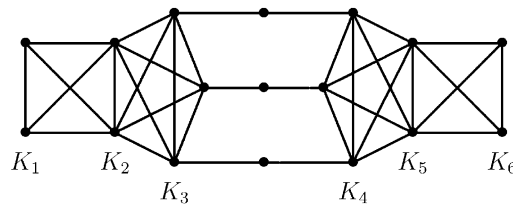


Fig. 4.

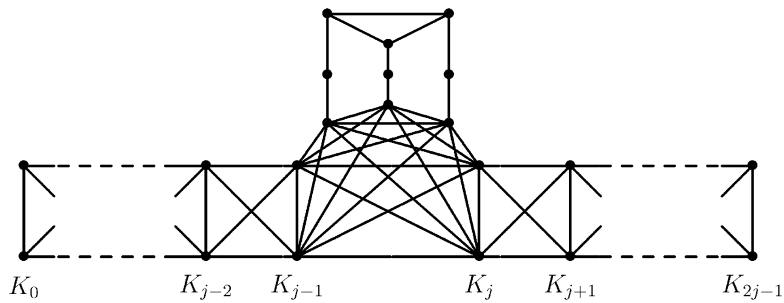


Fig. 5.

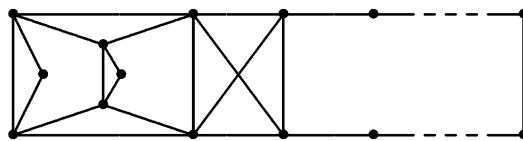


Fig. 6.

**Corollary 2.** *Let  $j \geq 2$  be an integer and let  $G$  be a 2-connected  $CN_{1,1,j}$ -free graph of diameter  $d \geq \max\{4, 2j\}$ . Then  $G$  is hamiltonian.*

The proofs of Theorem 1 and Corollary 2 are given in Section 3.

**Remarks.** (1) From [7] we know that every 2-connected  $CN_{1,1,1}$ -free graph is hamiltonian. The graph in Fig. 4 indicates that there are 2-connected nonhamiltonian graphs of diameter  $d = 6$  that are  $CN_{1,2,j}$ -free for any  $j \geq 2$ . The example in Fig. 5 shows that there are 2-connected nonhamiltonian graphs which are  $CN_{1,2,j}$ -free and have diameter  $d = 2j - 1$  for any  $j \geq 3$ . Hence the requirement  $d \geq \max\{7, 2j\}$  in Theorem 1 is sharp.

Moreover, for any  $j \geq 3$ , the graph in Fig. 6 is an example of a 2-connected nonhamiltonian  $CN_{1,3,j}$ -free graph of arbitrarily large diameter, and, similarly, the graph in Fig. 7 is 2-connected, nonhamiltonian,  $CN_{2,2,j}$ -free and also has arbitrarily large diameter. Hence the assumption  $CN_{1,2,j}$ -free in Theorem 1 is also best possible.



Fig. 7.

It is easy to see that, in fact, each of the examples in Figs. 4–7 yields an infinite family, since each of the vertical edges in the graphs of Figs. 4, 5 (marked in the figure by  $K_i$ ) and each of the edges incident with a vertex of degree 2 in the graphs of Figs. 6 and 7 can be blown up to a clique of arbitrary order.

(2) The graph in Fig. 5 is also an example of a nonhamiltonian  $CN_{1,1,j}$ -free graph of diameter  $d = 2j - 1$ . Moreover, the graph in Fig. 3 is a nonhamiltonian  $CN_{1,1,2}$ -free graph of diameter 3. Hence the assumption  $d \geq \max\{4, 2j\}$  of Corollary 2 is also best possible.

### 3. Proofs

Before we prove the main result of the paper, Theorem 1, we first make some preliminary observations on shortest paths and their neighborhoods. Their main idea can be found already in [7], however, for the sake of completeness, we include them here as well.

Let  $G$  be a claw-free graph, let  $x, y \in V(G)$  and let  $P: x = v_0 v_1 v_2 \cdots v_k = y$ ,  $k \geq 4$ , be a shortest  $xy$ -path in  $G$ . Let  $z \in V(G) \setminus V(P)$ .

1. If  $|N_P(z)| = 1$ , then, since  $G$  is claw-free,  $z$  is adjacent to  $x$  or to  $y$ .
2. If  $|N_P(z)| \geq 2$  and  $\{v_i, v_j\} \subset N_P(z)$ , then, since  $P$  is a shortest path,  $|i - j| \leq 2$ .
3. By (1) and (2),  $|N_P(z)| \leq 3$  for every vertex  $z \in V(G) \setminus V(P)$ . Moreover, the vertices of  $N_P(z)$  are consecutive on  $P$ .

This motivates the following notation:

$$N_i := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_{i-1}, v_i, v_{i+1}\}\} \quad \text{for } 1 \leq i \leq k-1,$$

$$M_i := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_{i-1}, v_i\}\} \quad \text{for } 1 \leq i \leq k,$$

$$M_0 := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_0\}\},$$

$$M_{k+1} := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_k\}\}.$$

Thus, by (1)–(3),  $N(P) \cup V(P) = (\bigcup_{i=1}^k N_i) \cup (\bigcup_{i=0}^{k+1} M_i) \cup V(P)$ .

We further denote  $S = V(P) \cup N(P)$ ,  $R = V(G) \setminus S$ ,  $M_i^* = N_R(M_i)$ ,  $1 \leq i \leq k$ , and  $N_i^* = N_R(N_i)$ ,  $1 \leq i \leq k-1$ .

The sets  $M_i$ ,  $N_i$ ,  $M_i^*$ ,  $N_i^*$  have the following properties.

**Lemma 3.** Let  $G$  be a claw-free graph,  $x, y \in V(G)$ , and let  $P: x = v_0 v_1 v_2 \cdots v_k = y$  ( $k \geq 3$ ) be a shortest  $x, y$ -path in  $G$ . Then

- (i)  $\langle N_i \cup \{v_i\} \rangle_G$  is complete for  $1 \leq i \leq k-1$  and  $\langle M_i \cup \{v_{i-1}, v_i\} \rangle_G$  is complete for  $1 \leq i \leq k$ ,
- (ii)  $\langle N_{i-1} \cup M_i \cup \{v_{i-1}, v_i\} \rangle_G$  and  $\langle M_i \cup N_i \cup \{v_{i-1}, v_i\} \rangle_G$  are complete for  $2 \leq i \leq k-1$ ,
- (iii)  $N_i^* = \emptyset$  for  $1 \leq i \leq k-1$ .

**Proof.** (i) If some  $N_i$  or  $M_i$  is not complete, then some  $v_j$ ,  $j \in \{i-1, i, i+1\}$ , is a center of an induced claw, a contradiction.

(ii) If  $x \in N_{i-1}$  and  $y \in M_i$  are nonadjacent, then  $\langle \{v_i, x, y, v_{i+1}\} \rangle_G$  is an induced claw. Hence  $\langle N_{i-1} \cup M_i \rangle_G$  (and similarly  $\langle M_i \cup N_i \rangle_G$ ) is also complete;  $v_{i-1}$  and  $v_i$  are adjacent to all its vertices.

(iii) If  $x \in N_i^*$ , then  $x$  has a neighbor  $y$  in  $N_i$  and  $\langle \{y, x, v_{i-1}, v_{i+1}\} \rangle_G$  is a claw.  $\square$

If, moreover,  $G$  is net-free, then we know more about its structure.

**Lemma 4.** Let  $j \geq 2$ , let  $G$  be a  $CN_{1,2,j}$ -free graph, let  $x, y \in V(G)$  and let  $P: x = v_0 v_1 v_2 \cdots v_k = y$  be a shortest  $xy$ -path of length  $k \geq \max\{7, 2j\}$  in  $G$ . Then

- (i)  $M_i^* = \emptyset$  for  $3 \leq i \leq k-2$ ,
- (ii) for every vertex  $z \in R$  we have  $N_P(z) = \emptyset$  and  $N_S(z) \subseteq M_0 \cup M_1 \cup M_2 \cup M_{k-1} \cup M_k \cup M_{k+1}$ .

**Proof.** (i) Suppose  $x \in M_i^*$  for some  $i$ ,  $3 \leq i \leq k-2$ , and let  $y$  be a neighbor of  $x$  in  $M_i$ . Then for  $i \leq \lfloor k/2 \rfloor$  we have  $\langle \{v_{i-3}, v_{i-2}, v_{i-1}, y, x, v_i, \dots, v_{i+j}\} \rangle_G \simeq N_{1,2,j}$ . The case  $i > \lfloor k/2 \rfloor$  is symmetric.

(ii)  $N_P(z) = \emptyset$  follows from the definition of  $R$ ; the rest follows from (3) of Lemma 3 and from (1) of Lemma 4.  $\square$

**Proof of Theorem 1.** Let  $G$  be a 2-connected  $CN_{1,2,j}$ -free graph of diameter  $d \geq \max\{7, 2j\}$ ,  $j \geq 2$ , and let  $P: v_0 v_1 v_2 \cdots v_d$  be a diameter path in  $G$ . Let  $M_i, N_i, M_i^*, S, R$  be as above. Set  $c = \lfloor d/2 \rfloor$ .

We distinguish two cases.

Case 1:  $M_c \cup N_c \cup \{v_c\}$  is not a cutset of  $G$ .

Let  $H = G - \{M_c \cup N_c \cup \{v_c\}\}$  and let  $P': v_d v_{d+1} v_{d+2} \cdots v_{d+\ell-1} v_{d+\ell} = v_0$  be a shortest  $v_d v_0$ -path in  $H$ . Since  $P$  is a diameter path,  $\ell \geq d$ . Since  $H$  is  $CN_{1,2,j}$ -free and  $P'$  is a shortest path in  $H$ , we can define analogously the sets  $M_i, N_i, M_i^*$  for  $i = d+1, \dots, d+\ell$ . Specifically, by (1) of Lemma 4 we have  $M_i^* = \emptyset$  for  $d+3 \leq i \leq d+\ell-2$ .

Set  $H' = G - \{M_2 \cup M_{d+\ell-2} \cup N_2 \cup N_{d+\ell-2} \cup \{v_2, v_{d+\ell-2}\}\}$ . By virtue of (1) of Lemma 4, there is a shortest  $v_3 v_{d+\ell-3}$ -path in  $H'$  containing all vertices  $v_i$  for  $i \in \{3, \dots, d-2\} \cup \{d+2, \dots, d+\ell-3\}$ . Let  $P_d: v_3 v_4 \cdots v_{d-2} P^d v_{d+2} \cdots v_{d+\ell-3}$  be such a path (where  $P^d$  is a  $v_{d-2} v_{d+2}$ -path). If  $v_{d-2} v_{d+2} \in E(G)$ , then  $v_{d+2} \in M_{d-1}$ , implying  $\langle \{v_{d+4}, v_{d+3}, v_{d+2},$

$v_{d-1}, v_d, v_{d-2}, \dots, v_{d-(j+2)}\rangle_G \simeq N_{1,2,j}$ , and if  $v_{d-2}v_{d+2}$  are at distance 2 with a common neighbor  $x$ , then similarly we have  $x \in M_{d-1}$  and  $\langle \{v_{d+3}, v_{d+2}, x, v_{d-1}, v_d, v_{d-2}, \dots, v_{d-(j+2)}\}\rangle_G \simeq N_{1,2,j}$ . Hence  $P^d$  is of length at least 3. Then the length of  $P_d$  is at least  $(d-5)+3+(\ell-5)=d+\ell-7 \geq \max\{7, 2j\}$  (since  $\ell \geq d$  and  $d \geq \max\{7, 2j\}$ ). Hence  $N(P^d)$  has the structure described in Lemmas 3 and 4.

Let, symmetrically,  $P_0 : v_{d+3}v_{d+4} \cdots v_{d+\ell-2}P^0v_2 \cdots v_{d-3}$  be a shortest  $v_{d+3}v_{d-3}$ -path in the graph  $H' = G - (M_{d-2} \cup M_{d+2} \cup N_{d-2} \cup N_{d+2} \cup \{v_{d-2}, v_{d+2}\})$  (where again  $P^0$  is a  $v_{d+\ell-2}v_2$ -path of length at least 1). Define the cycle  $C$  by  $C : v_2v_3 \cdots v_{d-2}P^dv_{d+2}v_{d+3} \cdots v_{d+\ell-2}P^0v_2$ . Then  $C$  is a (chordless) cycle of length  $\ell' \geq d-5+3+\ell-5+3 \geq 2d-4 > d$ . Relabel the vertices of  $P, P'$  such that  $C : v_1, v_2 \cdots v_{\ell'}v_1$ , and define the sets  $M_i, N_i, M_i^*, N_i^*$  accordingly (indices modulo  $\ell'$ ). Then  $M_i^* = N_i^* = \emptyset$  for all  $i = 1, \dots, \ell'$ , implying that  $(\bigcup_{i=1}^{\ell'} (M_i \cup N_i)) \cup V(C) = V(G)$ , i.e.,  $V(G) = V(C) \cup N(C)$ . Since clearly  $v_i$  is eligible in  $G$  if and only if  $N_i \neq \emptyset$  or  $E(M_i, M_{i+1}) \neq \emptyset$ , we conclude that  $\text{cl}(G)$  is either complete (if  $N_i = \emptyset$  or  $E(M_i, M_{i+1}) = \emptyset$  for at most three  $i = 1, \dots, \ell'$ ), or otherwise  $\text{cl}(G)$  consists of  $k$  cliques  $K_1, \dots, K_k$  ( $4 \leq k \leq \ell'$ ) such that  $|V(K_i) \cap V(K_j)| = 1$  for  $|i-j| \in \{1, k-1\}$  and  $V(K_i) \cap V(K_j) = \emptyset$  for  $i \neq j$  otherwise. In both cases, the hamiltonicity of  $\text{cl}(G)$  (and hence also of  $G$ ) is apparent.

Case 2:  $M_c \cup N_c \cup \{v_c\}$  is a cutset of  $G$ .

Then, by (1) of Lemma 4 and (2) of Lemma 3, each of  $\langle M_i \cup N_i \cup \{v_i\} \rangle_G$  is a clique cutset of  $G$  for  $i = 2, \dots, d-2$ .

First observe that  $M_0 \subset N(M_2)$  and  $M_1^* \subset M_2^*$ , for otherwise there is a vertex  $x \in M_0 \cup M_1^*$  at distance  $d+1$  from  $v_d$  (since every  $xv_d$ -path passes through  $M_2 \cup N_2 \cup \{v_2\}$ ). Symmetrically,  $M_{d+1} \subset N(M_{d-1})$  and  $M_d^* \subset M_{d-1}^*$ . Set

$$G_1 = \langle \{v_0, v_1, v_2\} \cup N_1 \cup N_2 \cup M_0 \cup M_1 \cup M_2 \cup M_2^* \rangle_G,$$

$$G_2 = \langle \{v_3, \dots, v_{d-3}\} \cup N_3 \cup \cdots \cup N_{d-3} \cup M_3 \cup \cdots \cup M_{d-2} \rangle_G,$$

$$G_3 = \langle \{v_{d-2}, v_{d-1}, v_d\} \cup N_{d-2} \cup N_{d-1} \cup M_{d-1} \cup M_d \cup M_{d+1} \cup M_{d-1}^* \rangle_G.$$

Since  $G$  is 2-connected, for any  $i = 2, \dots, d-2$  we have either  $N_i \neq \emptyset$ , or  $M_i \neq \emptyset$ ,  $M_{i+1} \neq \emptyset$  and  $E(M_i, M_{i+1}) \neq \emptyset$ . This implies that  $v_i$  is eligible in  $G$  for all  $i$ ,  $2 \leq i \leq d-2$ , and hence  $V(G_2)$  induces a clique in  $\text{cl}(G)$ .

If both  $V(G_1)$  and  $V(G_3)$  induce a clique in  $\text{cl}(G)$ , then  $\text{cl}(G)$  is complete and we are done. Hence, by symmetry, suppose that  $\langle V(G_1) \rangle_{\text{cl}(G)}$  is not complete. This immediately implies that  $v_1$  cannot be eligible in  $G$  (recall that  $M_0 \subset N(M_2)$  and  $M_1^* \subset M_2^*$ ), hence  $N_1 = \emptyset$  and  $E(M_1, M_2) = \emptyset$ . Since  $G$  is 2-connected,  $v_1$  cannot be a cutvertex, implying  $M_0 \neq \emptyset$  or  $M_1^* \neq \emptyset$ . Then it is straightforward to verify that in all possible cases (according to which of  $M_0, M_1^*$  is nonempty), for any  $v'_2 \in N_2 \cup M_2$  there is a  $v_2v'_2$ -path  $P_1$  such that  $V(P_1) = V(G_1)$  (note that  $\langle M_0 \rangle_G$  and all  $\langle N_R(v) \rangle_G$ ,  $v \in M_2$ , are complete since  $G$  is claw-free). By Theorem B(iv),  $G$  is hamiltonian.  $\square$

**Proof of Corollary 2.** Since  $N_{1,1,j}$ -free implies  $N_{1,2,j}$ -free, we immediately have the result for  $d \geq \max\{7, 2j\}$ . If  $G$  is  $N_{1,1,j}$ -free, then in Lemma 4(i) we moreover get  $M_i^* = \emptyset$  for  $2 \leq i \leq d-1$ , which, reconsidering the proof of Theorem 1, yields the



result for  $d \geq \max\{5, 2j\}$ . Finally, every  $CN_{1,1,2}$ -free graph of diameter at least 4 must be hamiltonian by Theorem E.  $\square$

## References

- [1] P. Bedrossian, Forbidden subgraph and minimum degree conditions for hamiltonicity, Thesis, Memphis State University, USA, 1991.
- [2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.
- [3] H.J. Broersma, H.J. Veldman, Restrictions on induced subgraphs ensuring hamiltonicity or pancyclicity of  $K_{1,3}$ -free graphs, in: R. Bodendiek (Ed.), Contemporary Methods in Graph Theory, BI-Wiss.-Verl, Mannheim-Wien-Zürich, 1990, pp. 181–194.
- [4] J. Brousek, Minimal 2-connected non-hamiltonian claw-free graphs, Discrete Math. 191 (1998) 57–64.
- [5] J. Brousek, O. Favaron, Z. Ryjáček, Forbidden subgraphs, hamiltonicity and closure in claw-free graphs, Discrete Math. 196 (1999) 29–50.
- [6] J. Brousek, Z. Ryjáček, I. Schiermeyer, Forbidden subgraphs, stability and hamiltonicity, Discrete Math. 197/198 (1999) 143–155.
- [7] D. Duffus, R.J. Gould, M.S. Jacobson, Forbidden subgraphs and the hamiltonian theme. The theory and applications of graphs, Kalamazoo, Mich 1980 and Wiley, New York, 1981, pp. 297–316.
- [8] R.J. Faudree, R.J. Gould, Characterizing forbidden pairs for hamiltonian properties, Discrete Math. 173 (1997) 45–60.
- [9] R.J. Faudree, Z. Ryjáček, I. Schiermeyer, Claw-free and generalized bull-free graphs of large diameter are hamiltonian, Tatra Mountains Math. Publ. 18 (1999) 105–113.
- [10] S. Goodman, S. Hedetniemi, Sufficient conditions for a graph to be hamiltonian, J. Combin. Theory Ser. B 16 (1974) 175–180.
- [11] R.J. Gould, M.S. Jacobson, Forbidden subgraphs and hamiltonian properties of graphs, Discrete Math. 42 (1982) 189–196.
- [12] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997) 217–224.